ASYMPTOTIC BEHAVIOR OF THE SOLUTION OF THE PROBLEM OF ELASTICITY FOR A THIN SPHERICAL SHELL

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The equilibrium of a symmetrically loaded, thin walled spherical shell is considered. A study is made of the asymptotic behavior of the solution as the relative thickness γ of the shell tends to zero. An asymptotic expansion has been established which estimates the error in the technical theory and enables one to derive more exact theories. Use is made of the method of homogeneous solutions, which in the case of spherical shells is due to Lur'e [1].

1. In order to construct homogeneous solutions for the spherical band (Fig.1) we write the equilibrium equations in the theory of elasticity in the system of coordinates r, α, φ

$$\frac{1}{1-2\nu}\frac{\partial\theta}{\partial r}+\nabla^2 u_r-\frac{2}{r^2}u_r-\frac{2}{r^2}\cosh\alpha\frac{\partial u_\alpha}{\partial\alpha}+\frac{2}{r^2}\sinh\alpha\, u_\alpha=0$$

$$\frac{1}{1-2\nu}\frac{\partial\theta}{\partial\alpha}+\frac{r}{\cosh^2\alpha}\nabla^2 u_\alpha+\frac{2}{r}\frac{\partial u_r}{\partial\alpha}-\frac{\cosh\alpha}{r}u_\alpha=0 \qquad (1.1)$$

$$\theta=\frac{\partial u_r}{\partial r}+\frac{2}{r}u_r+\frac{\cosh\alpha}{r}\frac{\partial u_\alpha}{\partial\alpha}-\frac{\sinh\alpha}{r}u_\alpha$$

$$\left(\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{\cosh^2\alpha}{r^2}\frac{\partial^2}{\partial\alpha^2}\right)\qquad\left(\alpha=\ln\tan\frac{\theta}{2}\right) \qquad (1.2)$$

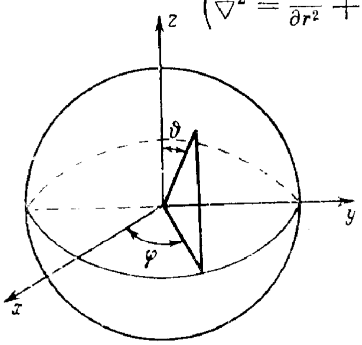

Fig. 1

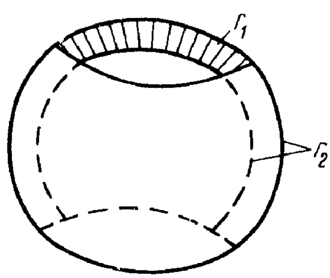

Fig. 2

The displacements u_r and u_α will be sought with the aid of the following relations

$$u_r = a(r)\, m(\alpha), \qquad u_\alpha = b(r) \cosh\alpha \frac{dm(\alpha)}{d\alpha}, \qquad \cosh^2\alpha \frac{d^2 m}{d\alpha^2} = \mu^2 m \tag{1.3}$$

The parameter μ in (1.3) is determined from the boundary conditions on the spherical parts Γ_2 of the boundaries of the band (Fig.2).

Substituting (1.2) into (1.1) and taking account of (1.3), it is found that a and b are governed by a system of two differential equations

$$\begin{aligned} a'' + \frac{2}{r} a' - \frac{3-2\nu}{2(1-\nu)} \frac{1}{r^2} a + \frac{1}{2(1-\nu)} \frac{\mu^2}{r} b' - \frac{3-4\nu}{1-2\nu} \frac{\mu^2}{r^2} b = 0 \\ b'' + \frac{2}{r} b' + \frac{2(1-\nu)}{1-2\nu} \frac{\mu^2}{r^2} b + \frac{1}{1-2\nu} \frac{1}{r} a' + \frac{4(1-\nu)}{1-2\nu} \frac{1}{r^2} a = 0 \end{aligned} \tag{1.4}$$

The system (1.4) is of the Euler type, and the general solution is easily written down

$$\begin{aligned} a(r) = -\frac{1}{4(1-2\nu)} \Big\{ [(t^2-4t-5) + 8\nu(t+1)] [r^{\frac{t+1}{2}} C_1 + r^{-\frac{t+3}{2}} C_2] + \\ + [(t^2+4t-5) - 8\nu(t-1)] [r^{-\frac{t-1}{2}} C_3 + r^{\frac{t-3}{2}} C_4] \Big\} \end{aligned} \tag{1.5}$$

$$\begin{aligned} b(r) = -\frac{1}{2(1-2\nu)} \Big\{ [(t+9) - 8\nu] r^{\frac{t+1}{2}} C_1 - [(t-5) + 8\nu] r^{-\frac{t+3}{2}} C_2 - \\ - [(t-9) + 8\nu] r^{-\frac{t-1}{2}} C_3 + [(t+5) - 8\nu] r^{\frac{t-3}{2}} C_4 \Big\} \quad (t = \sqrt{1-4\mu^2}) \end{aligned}$$

By using (1.5), it is easy to find the formulas for $u_r, u_\alpha, \tau_{r\alpha}, \sigma_r, \sigma_\alpha, \sigma_\varphi$

$$\begin{aligned} u_r = -\frac{1}{4(1-2\nu)} \Big\{ [(t^2-4t-5) + 8\nu(t+1)] [r^{\frac{t+1}{2}} C_1 + r^{-\frac{t+3}{2}} C_2] + \\ + [(t^2+4t-5) - 8\nu(t-1)] [r^{-\frac{t-1}{2}} C_3 + r^{\frac{t-3}{2}} C_4] \Big\} m(\alpha) \end{aligned} \tag{1.6}$$

$$\begin{aligned} u_\alpha = -\frac{1}{2(1-2\nu)} \Big\{ [(t+9) - 8\nu] r^{\frac{t+1}{2}} C_1 - [(t-5) + 8\nu] r^{-\frac{t+3}{2}} C_2 - \\ - [(t-9) + 8\nu] r^{-\frac{t-1}{2}} C_3 + [(t+5) - 8\nu] r^{\frac{t-3}{2}} C_4 \Big\} \cosh\alpha \frac{dm(\alpha)}{d\alpha} \end{aligned}$$

$$\begin{aligned} \tau_{r\alpha} = -\frac{E}{4(1+\nu)(1-2\nu)} \Big\{ [(t^2+2t-7) + 8\nu] r^{\frac{t-1}{2}} C_1 + (t+3)[(t-5) + 8\nu] r^{-\frac{t+5}{2}} C_2 + \\ + [(t^2-2t-7) + 8\nu] r^{-\frac{t+1}{2}} C_3 + (t-3)[(t+5) - 8\nu] r^{\frac{t-5}{2}} C_4 \Big\} \cosh\alpha \frac{dm(\alpha)}{d\alpha} \end{aligned} \tag{1.7}$$

$$\begin{aligned} \sigma_r = -\frac{E}{8(1+\nu)(1-2\nu)} \Big\{ (t+1)[(t^2-4t-5) - 8\nu] r^{\frac{t-1}{2}} C_1 - \\ - (t^2+4t+3)[(t-5) + 8\nu] r^{-\frac{t+5}{2}} C_2 - \\ - (t-1)[(t^2+4t-5) - 8\nu] r^{-\frac{t+1}{2}} C_3 + (t^2-4t+3)[(t+5) - 8\nu] r^{\frac{t-5}{2}} C_4 \Big\} m(\alpha) \end{aligned}$$

$$\sigma_\alpha = \frac{E}{8(1+\nu)(1-2\nu)} \Big\{ \Big\langle (t+1)\,[(t^2+6t+1)+8\nu]\, r^{\frac{t-1}{2}} C_1 + (t+1)^2 \times$$
$$\times [(-t+5)-8\nu]\, r^{-\frac{t+5}{2}} C_2 - (t-1)\,[(t^2-6t+1)+8\nu]\, r^{-\frac{t+1}{2}} C_3 + (t-1)^2 \times$$
$$\times [(t+5)-8\nu]\, r^{\frac{t-5}{2}} C_4 \Big\rangle m(\alpha) - 2\Big\langle [(t+9)-8\nu]\, r^{\frac{t-1}{2}} C_1 - [(t-5)+8\nu]\, r^{-\frac{t+5}{2}} C_2 -$$
$$- [(t-9)+8\nu]\, r^{-\frac{t+1}{2}} C_3 + [(t+5)-8\nu]\, r^{\frac{t-5}{2}} C_4 \Big\rangle \sinh 2\alpha\, \frac{dm(\alpha)}{d\alpha} \Big\} \qquad (1.8)$$

$$\sigma_\varphi = \frac{E}{4(1+\nu)(1-2\nu)} \Big\{ \Big\langle (t+1)\,[(-t+5)+4\nu t]\, r^{\frac{t-1}{2}} C_1 + (t+1) \times$$
$$\times [(-t+5)-8\nu]\, r^{-\frac{t+5}{2}} C_2 - (t-1)\,[(t+5)-4\nu t]\, r^{-\frac{t+1}{2}} C_3 -$$
$$- (t-1)\,[(t+5)-8\nu]\, r^{\frac{t-5}{2}} C_4 \Big\rangle m(\alpha) +$$
$$+ \Big\langle [(t+9)-8\nu]\, r^{\frac{t-1}{2}} C_1 - [(t-5)+8\nu]\, r^{-\frac{t+5}{2}} C_2 - [(t-9)+8\nu]\, r^{-\frac{t+1}{2}} C_3 +$$
$$+ [(t+5)-8\nu]\, r^{\frac{t-5}{2}} C_4 \Big\rangle \sinh 2\alpha\, \frac{dm(\alpha)}{d\alpha} \Big\}$$

The quantities C_i are determined by the homogeneity condition, in view of which we have

$$\sigma_r\big|_{r=a} = -(t+1)\,[(t^2-4t-5)-8\nu]\, a^{\frac{t-1}{2}} C_1 + (t^2+4t+3)\,[(t-5)+8\nu]\, a^{-\frac{t+5}{2}} C_2 +$$
$$+ (t-1)\,[(t^2+4t-5)-8\nu]\, a^{-\frac{t+1}{2}} C_3 - (t^2-4t+3)\,[(t+5)-8\nu]\, a^{\frac{t-5}{2}} C_4 = 0$$

$$\tau_{r\alpha}\big|_{r=a} = [(t^2+2t-7)+8\nu]\, a^{\frac{t-1}{2}} C_1 + (t+3)\,[(t-5)+8\nu]\, a^{-\frac{t+5}{2}} C_2 +$$
$$+ [(t^2-2t-7)+8\nu]\, a^{-\frac{t+1}{2}} C_3 + (t-3)\,[(t+5)-8\nu]\, a^{\frac{t-5}{2}} C_4 = 0$$

$$\sigma_r\big|_{r=b} = -(t+1)\,[(t^2-4t-5)-8\nu]\, b^{\frac{t-1}{2}} C_1 + (t^2+4t+3)\,[(t-5)+8\nu]\, b^{-\frac{t+5}{2}} C_2 +$$
$$+ (t-1)\,[(t^2+4t-5)-8\nu]\, b^{-\frac{t+1}{2}} C_3 - (t^2-4t+3)\,[(t+5)-8\nu]\, b^{\frac{t-5}{2}} C_4 = 0$$

$$\tau_{r\alpha}\big|_{r=b} = [(t^2+2t-7)+8\nu]\, b^{\frac{t-1}{2}} C_1 + (t+3)\,[(t-5)+8\nu]\, b^{-\frac{t+5}{2}} C_2 + \qquad (1.9)$$
$$+ [(t^2-2t-7)+8\nu]\, b^{-\frac{t+1}{2}} C_3 + (t-3)\,[(t+5)-8\nu]\, b^{\frac{t-5}{2}} C_4 = 0$$

Setting the determinant of system (1.9) equal to zero, we obtain the following equation for $\beta = \frac{1}{2}t$:

$$\left(\frac{\lambda^\beta - \lambda^{-\beta}}{\lambda - \lambda^{-1}}\right)^2 = \beta^2\, \frac{\beta^4 - {}^5/_2\,\beta^2 + {}^{73}/_{16} - 4\nu^2}{\beta^4 + \beta^2\,[4(1-\nu^2) - {}^5/_2] + {}^9/_{16}} \qquad \left(\lambda = \frac{b}{a}\right) \qquad (1.10)$$

Equation (1.10) was first derived in [1]. From system (1.9) the C_i are found to be

$$
\begin{aligned}
C_1 &= 2\,(t^2-9)\,[(t^2-25)+80\nu-64\nu^2]\,a^{-\frac{t+11}{2}}\,\{[t^3-7t+6]\,\lambda^{-3}-\\
&\quad-[(t^3-2t^2-7t)+8\nu t]\,\lambda^{-5}-2\,[(t^2+3)-4\nu t]\,\lambda^{-(t+3)}\}\\
C_2 &= 2\,(t-3)\,[(t+5)-8\nu]\,a^{\frac{t-7}{2}}\,\{[(t^5-10t^3+73t)-64\nu^2 t]\,\lambda^{-1}-\\
&\quad-(t^3-7t+6)\,[(t^2+2t-7)+8\nu]\,\lambda^{-3}+2\,[(t^4-2t^3-4t^2-6t-21)+\\
&\quad+4\nu\,(t^3-7t+6)+32\nu^2 t]\,\lambda^{t-3}\}\\
C_3 &= 2\,(t^2-9)\,[(t^2-25)+80\nu-64\nu^2]\,a^{\frac{t-11}{2}}\,\{-[t^3-7t-6]\,\lambda^{-3}+\\
&\quad+[(t^3+2t^2-7t)+8\nu t]\,\lambda^{-5}-2\,[(t^2+3)+4\nu t]\,\lambda^{t-3}\}\\
C_4 &= -2\,(t+3)\,[(t-5)+8\nu]\,a^{-\frac{t+7}{2}}\,\{[(t^5-10t^3+73t)-64\nu^2 t]\,\lambda^{-1}-\\
&\quad-(t^3-7t-6)\,[(t^2-2t-7)+8\nu]\,\lambda^{-3}-2\,[(t^4+2t^3-4t^2+6t-21)-\\
&\quad-4\nu\,(t^3-7t-6)-32\nu^2 t]\,\lambda^{-(t+3)}\}
\end{aligned}
\tag{1.11}
$$

Equation (1.10) can be transformed into a form more convenient for further investigations

$$
\left(\frac{\sinh\gamma\beta}{\sinh\gamma}\right)^2=\beta^2 f(\beta);\quad \gamma=\ln\lambda,\quad f(\beta)=\frac{\beta^4-{}^5\!/_2\,\beta^2+{}^{73}\!/_{16}-4\nu^2}{\beta^4+\beta^2\,[4\,(1-\nu^2)-{}^5\!/_2]+{}^9\!/_{16}} \tag{1.12}
$$

2. In this section we will study the roots of Equation (1.12). First, by direct substitution one can verify that (1.12) has the three real roots $\beta = 0$ and $\beta = \pm 1$. Then it is not difficult to establish that it has no further real roots. In fact, when $\beta > 0$ the function $(\sinh \gamma\beta/\beta \sinh \gamma)^2$ is monotonously increasing, and $f(\beta)$ is monotonously decreasing. Consequently, when $\beta > 0$ there can be at most one intersection of their two graphs. This point is $\beta = 1$. From the evenness of Equation (1.12), it follows that the above assertion is correct.

Now let $\beta_k(\gamma)$ be an arbitrary complex root of Equation (1.12). We will prove that $\beta_k \to \infty$ as $\gamma \to 0$.

First we note that the complex root β_k cannot tend to zero as $\gamma \to 0$. In fact, if this were so, we would obviously have

$$
\left(\frac{\sinh\gamma\beta}{\beta\sinh\gamma}\right)^2\to 1,\qquad f(\beta)\to\frac{{}^{73}\!/_{16}-4\nu^2}{{}^9\!/_{16}}\neq 1
$$

Thus, we can choose a sequence β_k which tends to the finite limit $\beta^{(0)}$, as $\gamma \to 0$. Then, however, it follows by virtue of (1.12) that $f(\beta^{(0)}) = 1$, as $\gamma \to 0$, which in turn implies that $\beta^{(0)} = \pm 1$. We will prove that this is also impossible. In fact, when $\gamma \to 0$ and $(\beta^{(0)})^2 \to 1$ we have

$$
\left(\frac{\gamma\beta+{}^1\!/_6\gamma^3\beta^3+\ldots}{\gamma+{}^1\!/_6\gamma^3+\ldots}\right)^2=\beta^2 f(\beta),\qquad\text{or}\qquad \frac{1+{}^1\!/_3\gamma^2\beta^2+\ldots}{1+{}^1\!/_3\gamma^2+\ldots}=f(\beta) \tag{2.1}
$$

From (2.1) we obtain

$$
\frac{{}^1\!/_3\gamma^2\,(\beta^2-1)+\ldots}{1+{}^1\!/_3\gamma^2+\ldots}=\frac{-4\,(1-\nu^2)\,(\beta^2-1)}{\beta^4+\beta^2\,[4\,(1-\nu^2)-{}^5\!/_2]+{}^9\!/_{16}} \tag{2.2}
$$

Cancelling the factor $(\beta^2 - 1)$ in (2.2) and letting γ tend to zero, we obtain a contradiction. Thus we have proved that $\beta_k \to \infty$ as $\gamma \to 0$.

Now we examine in what way β_k becomes infinite as $\gamma \to 0$. Let us consider the expression $\gamma\beta_k$. As $\gamma \to 0$ there are three possible cases

$$1)\quad \gamma\beta_k \to \text{const} < \infty; \quad 2)\quad \gamma\beta_k \to 0; \quad 3)\quad \gamma\beta_k \to \infty$$

We will show that the third case leads to a contradiction. Since $\beta_k \to \infty$ as $\gamma \to 0$, then $f(\beta_k) \to 1$. It then follows from (1.12) that we must have $\sinh^2\gamma\beta_k \sim (\gamma\beta_k)^2$, which is impossible for continuous variation of $\gamma\beta_k$.

Now we will consider the first case. We will denote the finite limit to which $\gamma\beta_k$ tends as $\gamma \to 0$ by $m_{-1}^{(k)}$. Then it is easy to see from (1.12) that $m_{-1}^{(k)}$ satisfies Equation

$$\sinh^2 m_{-1}^{(k)} - (m_{-1}^{(k)})^2 = 0 \tag{2.3}$$

and $\beta_k \to \infty$ like $m_{-1}^{(k)} / \gamma$. It is important to note that Equation (2.3) is actually identical with the equation that determines the exponents in the Saint-Venant boundary effects in the theory of plates [2 and 3]. Equation (2.3) has a denumerable set of roots, therefore, it follows that Equation (1.12) has also denumerable set of roots, such as $\gamma\beta_k \to$ const. It is easy to refine the value of the above roots by using the expansion

$$\beta_k = \frac{m_{-1}^{(k)}}{\gamma} + m_1^{(k)}\gamma + m_3^{(k)}\gamma^3 + \ldots \tag{2.4}$$

and

$$m_1^{(k)} = \frac{1}{3}\,\frac{12(1-\nu^2) - (m_{-1}^{(k)})^2}{\sinh 2m_{-1}^{(k)} - 2m_{-1}^{(k)}} \quad \text{etc.}$$

Now we take up the study of the roots in the second group, for which $\gamma\beta_k \to 0$ as $\gamma \to 0$. Let us denote $\gamma\beta_k$ by x_k. Equation (1.12) can then be represented in the form

$$\begin{aligned} F(x_k, \gamma) = \gamma^2 \sinh^2 x_k\, [x_k^4 + x_k^2 (3/2 - 4\nu^2)\gamma^2 + 9/16\,\gamma^4] - \\ - \sinh^2 \gamma x_k^2\, [x_k^4 - 5/2\, x_k^2\gamma^2 + (73/16 - 4\nu^2)\gamma^4] = 0 \end{aligned} \tag{2.5}$$

When x_k and γ are small, function $F(x_k, \gamma)$ can be expanded in a power series so that Equation (2.5) becomes

$$\begin{aligned} [x_k^6 + 2/15\, x_k^8 + 1/105\, x_k^{10} + 1/4725\, x_k^{12} + \ldots] + 3\,[4(1-\nu^2)x_k^2 + \\ + 1/6\,(1-8\nu^2)x_k^4 + 1/45\,(3-8\nu^2)x_k^6 + 1/630\,(3-8\nu^2)x_k^8 + \\ + 1/28350\, x_k^{10} + \ldots]\gamma^2 - 3\,[4(1-\nu^2) - 49/48\, x_k^2 + 7/360\, x_k^4 - 1/560\, x_k^6 - \\ - 1/25200\, x_k^8 + \ldots]\gamma^4 - [(73/16 - 4\nu^2) - 1/3\, x_k^2 + 1/315\, x_k^4 + \ldots]\gamma^6 - \\ - 3\,[2/45\,(73/16 - 4\nu^2) - 1/126\, x_k^2 + 1/14175\, x_k^4 + \ldots]\gamma^8 + \ldots = 0 \end{aligned} \tag{2.6}$$

If follows from (2.6) that

$$x_k = a_{-1}^{(k)}\gamma^{1/2} + a_1^{(k)}\gamma^{3/2} + a_3^{(k)}\gamma^{5/2} + \ldots$$

$$\beta_k = \frac{a_{-1}^{(k)}}{\gamma^{1/2}} + a_1^{(k)}\gamma^{1/2} + a_3^{(k)}\gamma^{3/2} + \ldots \tag{2.7}$$

and further

$$(a_{-1}^{(k)})^4 + 12(1-\nu^2) = 0, \quad a_1^{(k)} = \frac{1+24\nu^2}{40a_{-1}^{(k)}}$$

$$a_3^{(k)} = -\frac{9927 - 464\nu^2 - 10688\nu^4}{22400\,(a_{-1}^{(k)})^3} \quad \text{etc.} \tag{2.8}$$

It is clear from (2.8) that the second group contains four roots.

The above analysis shows that the characteristic Equation (1.12) has three groups of roots:

1. Roots independent of γ, namely, $\beta = 0$, $\beta = \pm 1$.
2. Four roots that increase like $\gamma^{-\frac{1}{2}}$ as $\gamma \to 0$.
3. A denumerable set of roots that increase like $1/\gamma$ as $\gamma \to 0$.

3. Now we will analyze the states of stress and strain corresponding to each of these groups of roots.

First group. For the root $\beta = 0$ we have

$$a(r) = (5-8\nu)[r^{1/2}C_1 + r^{-3/2}C_2] + [8(1-2\nu) + (5-8\nu)\ln r]\, r^{1/2}C_3 + \\ + [8(1-2\nu) - (5-8\nu)\ln r]\, r^{-3/2}C_4 \tag{3.1}$$

$$b(r) = -2\{(9-8\nu)r^{1/2}C_1 - (5-8\nu)r^{-3/2}C_2 \quad [2 + (9-8\nu)\ln r]\, r^{1/2}C_3 - \\ - [2 - (5-8\nu)\ln r]\, r^{-3/2}C_4\}$$

Substituting (3.1) into the boundary conditions $\sigma_r = \tau_{r\alpha} = 0$ when $r = a, b$, we obtain a system of four equations for the determination of C_i. The calculations show that $C_1 = C_2 = C_3 = C_4 = 0$.

For the roots $\beta = \pm 1$, we have

$$a(r) = 3(3-8\nu)[r^{3/2}C_1 + r^{-3/2}C_2] - (7-8\nu)[r^{-1/2}C_3 + r^{-1/2}\ln r\, C_4]$$

$$b(r) = -2\{(11-8\nu)r^{3/2}C_1 + (3-8\nu)r^{-3/2}C_2 + (7-8\nu)r^{-1/2}C_3 + \\ + [2 + (7-8\nu)\ln r]\, r^{-1/2}C_4\} \tag{3.2}$$

The analogous calculations to those in the preceding case yield $C_1 = C_2 = C_3 = C_4 = 0$.

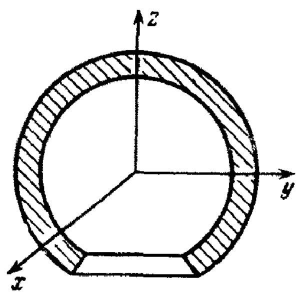

Fig. 3

Thus, the roots of the first group determine a state of stress that vanishes identically.

Second group. The equation for the determination of $m(\alpha)$ in this case has the form

$$\cosh^2\alpha\, d^2m / d\alpha^2 = \mu^2 m \tag{3.3}$$

where, in view of (1.5), (1.10) and (2.7), μ has the expansion

$$\mu = \frac{a_{-1}}{\sqrt{\gamma}}\, i\left\{1 + \frac{8a_{-1}a_1 - 1}{8a_{-1}^2}\gamma + \ldots\right\} \tag{3.4}$$

The solution of Equation (3.3) can, in general, be written in terms of Legendre functions. However, for the following it is more convenient to make use of an approximate method. Moreover, it proves to be expedient to consider the following two cases separately:

1) The shell does not contain either of the poles $\vartheta = 0, \pi$;
2) The shell contains at least one of these poles.

In the first case the approximate integration is conveniently carried out using the asymptotic method that has been explained in detail in [1]. Here we merely quote the final result

$$m(\alpha) = \exp\left\{ \tan^{-1}(\sinh\alpha)\frac{a_{-1}}{\sqrt{\gamma}} i + {}^1/_2 \cosh\alpha + \right. \tag{3.5}$$
$$\left. + \frac{1}{8a_{-1}}\left[\tan^{-1}(\sinh\alpha)(8\,a_{-1}\,a_1 - 1)i - \left(\frac{\sinh 2\alpha}{2} + \alpha + \frac{\sinh^3\alpha}{3}\right)\right]\sqrt{\gamma} + \ldots \right\}$$

It follows from (3.5) that, for sufficiently small γ, the quantity $m(\alpha)$ has the character of a boundary effect that varies as an exponental function with the index $\gamma^{-\frac{1}{2}}$. Thus, the second group of roots determines a boundary effect having the typical rate of decrease familiar in the technical theory of plates.

Now we will consider the second case (Fig.3). Here the usual asymptotic method of integration cannot give an adequate approximate solution irrespective of the relative thickness γ of the shell. The fact is that the asymptotic approximation loses accuracy in the vicinity of the pole $\vartheta = 0$. We note that in the present case it is necessary to select from the solutions of Equation (3.3) only those solutions that remain bounded when $\vartheta = 0$. There are two such solutions. These solutions were actually constructed in [1], where one will also find an approximate method for calculating them. We will simply quote the final result. We will denote the solution equal to 1 when $\vartheta = 0$, by $m(\cos\vartheta)$.

For small values of ϑ we have

$$m(\cos\vartheta) = J_0(\beta_k\vartheta) + [{}^1/_{46}\, J_0(\beta_k\vartheta) - {}^1/_{48} J_2(\beta_k\vartheta)]\vartheta^2 + \ldots \tag{3.6}$$

In connection with Formula (3.6), one important feature should be noted. In the present case when γ is small m behaves in the following manner. In the region $\vartheta = \vartheta_0$ it decreases and behaves like a boundary layer over the main range of ϑ. However, in the vicinity of $\vartheta = 0$ the quantity m again begins to increase and we have something like an inner boundary effect in the vicinity of the pole $\vartheta = 0$ $(\alpha = \infty)$ (Fig.4).

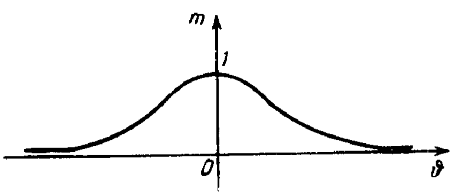

Fig. 4

For the roots of the second group, the formulas for the calculation of the stresses and displacements, when represented as power series in γ, have the following form

(3.7)

$$u_r^{(0)} = \frac{1}{1-2\nu}\Big\{2a_{-1}^2(1-\nu)\frac{1}{\sqrt{\gamma}} - \Big\langle 3\nu(1-\nu^2)\zeta^2 + \nu\,[a_{-1}^2(1-\nu)+12(1-\nu^2)]\,\zeta -$$

$$-\Big[(16a_{-1}a_1 - 11 - 7a_{-1}^2) - \nu(16a_{-1}a_1 - 42 - 8a_{-1}^2) - 2\nu^2\Big(46 + \frac{a_{-1}^2}{2}\Big) + 49\nu^3\Big]\Big\rangle\sqrt{\gamma} + \ldots\Big\}m$$

$$u_\alpha^{(0)} = -\frac{1}{1-2\nu}\,\{[a_{-1}^2(1-\nu)\zeta - 2(1-\nu^2)]\,\sqrt{\gamma} + \ldots\}\cosh\alpha\,\frac{dm}{d\alpha} \tag{3.8}$$

$$\tau_{r\alpha}^{(0)} = \frac{2G}{1-2\nu}\,a^{-1}\{3(1-\nu^2)(\zeta^2-1)\,\sqrt{\gamma} + \ldots\}\cosh\alpha\,\frac{dm}{d\alpha} \tag{3.9}$$

$$\sigma_r^{(0)} = \frac{2G}{1-2\nu}\,a^{-1}\{3(1-\nu^2)(\zeta^2-1)\,[{}^1\!/_6\,\zeta^2 a_{-1}^2 - (1+\nu)]\,\sqrt{\gamma} + \ldots\}\,m \tag{3.10}$$

$$\sigma_\alpha^{(0)} = -\frac{2G}{1-2\nu}\,a^{-1}\Big\{\Big[12(1-\nu^2)\zeta\frac{1}{\sqrt{\gamma}} + \ldots\Big]m +$$

$$+\langle[a_{-1}^2(1-\nu)\zeta - 2(1-\nu^2)]\,\sqrt{\gamma}\rangle\frac{\sinh 2\alpha}{2}\frac{dm}{d\alpha}\Big\} \tag{3.11}$$

$$\sigma_\varphi^{(0)} = -\frac{2G}{1-2\nu}\,a^{-1}\Big\{\Big\langle[12(1-\nu^2)\nu\zeta - 2a_{-1}^2(1-\nu^2)]\frac{1}{\sqrt{\gamma}} + \ldots\Big\rangle m -$$

$$-\langle[a_{-1}^2(1-\nu)\zeta - 2(1-\nu^2)]\,\sqrt{\gamma} + \ldots\rangle\frac{\sinh 2\alpha}{2}\frac{dm}{d\alpha}\Big\} \tag{3.12}$$

where ζ is the current coordinate measured from the middle surface

$$r = {}^1\!/_2\,a\,[(1-\zeta) + (1+\zeta)\,e^\gamma] \qquad (-1 \leqslant \zeta \leqslant 1) \tag{3.13}$$

In order to get a picture of the state of stress corresponding to the present group of roots we calculate the resultant force and moment of the stresses acting on the cross-section α = const On the basis of (1.12), it is easily verified that

$$P = \int_a^b \Big(\sigma_\alpha\frac{1}{\cosh\alpha} + \tau_{r\alpha}\tanh\alpha\Big)\,r\,dr = 0$$

$$M = \int_a^b \sigma_\alpha r^2 dr \sim [(1-\nu^2)\,a_{-1}\gamma^{3/2} + \ldots]\sinh 2\alpha\,\frac{dm}{d\alpha}$$

Thus the state of stress for this group has a resultant with zero component along the symmetry axis, and the resultant moment on elements of area in the cross-section α = const has order of smallness γ .

T h i r d g r o u p o f r o o t s . The equation for the determination of $m(\alpha)$ still has the form (3.3), but μ is given by Formula

$$\mu = \frac{m_{-1}}{\gamma}\,i\Big\{1 + \frac{8m_{-1}m_1 - 1}{8m_{-1}^2}\gamma^2 + \ldots\Big\} \tag{3.14}$$

Everything that has been said about the integration of equation (3.3) for the roots of the second group applies in the case of the third group of roots. In the present case

$$m(\alpha) = \exp\Big\{\tan^{-1}(\sinh\alpha)\frac{m_{-1}}{\gamma}i + \frac{1}{2}\cosh\alpha +$$

$$+\frac{1}{8m_{-1}}\Big[\tan^{-1}(\sinh\alpha)(8m_{-1}m_1 - 1) - \Big(\frac{\sinh 2\alpha}{2} + \alpha + \frac{\sinh^3\alpha}{3}\Big)\Big]\gamma + \ldots\Big\} \tag{3.15}$$

It is evident from (3.15) that, for sufficiently small γ, the function $m(\alpha)$ also has the character of a boundary effect varying as an exponential function with index $1/\gamma$. This fact distinguishes the Saint-Venant boundary effect from the boundary effect in the technical theory of shells.

In the present case the stresses and displacements can also be represented in the form of a series expansion in powers of γ. Below, in the first approximation, we will write out separately the expansions for β_k which degenerate as $\gamma \to 0$ into the roots of the equations $\sinh 2\omega_k - 2\omega_k = 0$ and into the roots of the equation $\sinh 2\delta_k + 2\delta_k = 0$. They have the following form

For the first case

$$u_{rk}^{(c)} = \frac{1}{1-2\nu}\{[2(1-\nu)\cosh\omega_k + \omega_k \sinh\omega_k]\cosh\omega_k\zeta - \zeta\omega_k\cosh\omega_k\sinh\omega_k\zeta\}\, m \qquad (3.16)$$

$$u_{\alpha k}^{(c)} = \frac{-1}{2(1-2\nu)}\left\{\left[(1-2\nu)\frac{\cosh\omega_k}{\omega_k} - \sinh\omega_k\right]\sinh\omega_k\zeta + \zeta\cosh\omega_k\cosh\omega_k\zeta\right\}\gamma\cosh\alpha\,\frac{dm}{d\alpha} \qquad (3.17)$$

$$\tau_{r\alpha k}^{(c)} = \frac{2G}{1-2\nu}a^{-1}\{[\sinh\omega_k\cosh\omega_k\zeta - \zeta\cosh\omega_k\sinh\omega_k\zeta]\,\omega_k\}\cosh\alpha\,\frac{dm}{d\alpha} \qquad (3.18)$$

$$\sigma_{rk}^{(c)} = -\frac{4G}{1-2\nu}a^{-1}\left\{\left[\left(\frac{\cosh\omega_k}{\omega_k} + \sinh\omega_k\right)\sinh\omega_k\zeta - \zeta\cosh\omega_k\cosh\omega_k\zeta\right]\omega_k^2\right\}\frac{1}{\gamma}m \qquad (3.19)$$

$$\sigma_{\alpha k}^{(c)} = \frac{4G}{1-2\nu}a^{-1}\left\{\left[\left(\frac{\cosh\omega_k}{\omega_k} - \sinh\omega_k\right)\sinh\omega_k\zeta + \zeta\cosh\omega_k\cosh\omega_k\zeta\right]\omega_k^2\right\}\frac{1}{\gamma}m \qquad (3.20)$$

$$\sigma_{\varphi k}^{(c)} = \frac{8G}{1-2\nu}a^{-1}\nu\cosh\omega_k\sinh\omega_k\zeta\omega_k\,\frac{1}{\gamma}m \qquad (3.21)$$

For the second case

$$u_{rk}^{*(c)} = \frac{1}{1-2\nu}\{[2(1-\nu)\sinh\delta_k + \delta_k\cosh\delta_k]\sinh\delta_k\zeta - \zeta\delta_k\sinh\delta_k\cosh\delta_k\zeta\}\, m \qquad (3.22)$$

$$u_{\alpha k}^{*(c)} = -\frac{1}{2(1-2\nu)}\left\{\left[(1-2\nu)\frac{\sinh\delta_k}{\delta_k} - \cosh\delta_k\right]\cosh\delta_k\zeta + \zeta\sinh\delta_k\sinh\delta_k\zeta\right\}\gamma\cosh\alpha\,\frac{dm}{d\alpha} \qquad (3.23)$$

$$\tau_{r\alpha k}^{*(c)} = \frac{2G}{1-2\nu}a^{-1}\{[\cosh\delta_k\sinh\delta_k\zeta - \zeta\sinh\delta_k\cosh\delta_k\zeta]\,\delta_k\}\cosh\alpha\,\frac{dm}{d\alpha} \qquad (3.24)$$

$$\sigma_{rk}^{*(c)} = -\frac{4G}{1-2\nu}a^{-1}\left\{\left[\left(\frac{\sinh\delta_k}{\delta_k} + \cosh\delta_k\right)\cosh\delta_k\zeta - \zeta\sinh\delta_k\sinh\delta_k\zeta\right]\delta_k^2\right\}\frac{1}{\gamma}m \qquad (3.25)$$

$$\sigma_{\alpha k}^{*(c)} = \frac{4G}{1-2\nu}a^{-1}\left\{\left[\left(\frac{\sinh\delta_k}{\delta_k} - \cosh\delta_k\right)\cosh\delta_k\zeta + \zeta\sinh\delta_k\sinh\delta_k\zeta\right]\delta_k^2\right\}\frac{1}{\gamma}m \qquad (3.26)$$

$$\sigma_{\varphi k}^{*(c)} = \frac{8G}{1-2\nu}a^{-1}\nu\sinh\delta_k\cosh\delta_k\zeta\delta_k\,\frac{1}{\gamma}m \qquad (3.27)$$

It is clear that in the first case u_r is an even function of ζ, and ~ is odd, which corresponds to a predominant flexure in the shell. In the second case the solutions correspond to a predominance of membrane stressing. It is important to note that as $\gamma \to 0$ the boundary effects caused by the present group of roots go over exactly into the Saint-Venant boundary effect in the theory of plates.

Now we will calculate the resultant force and moment of the stresses acting on the cross-section $\alpha = \text{const}$. In view of (1.12) we find the resultant force has no component along the symmetry axis and that the resultant moment on the elements of surface have the order of smallness γ^2, namely

$$M = \int_a^b \sigma_\alpha r^2 dr \sim \{(1-\cosh m_{-1})[\sinh m_{-1} - m_{-1}]\gamma^2 + \ldots\}\left[4m(\alpha) - \frac{\sinh 2\alpha}{2}\gamma^2\frac{dm(\alpha)}{d\alpha}\right]$$

The present system of stresses can be assumed equivalent to zeı with a degree of accuracy γ^2. Thus, the homogeneous solutions defined by the roots of the second and third groups can be used to balance the stresses arising in a shell under the action of a self-equilibrating system of forces applied to the surfaces α = const . When the transverse edges of the shell are subjected to forces whose resultant has a nonzero component along the symmetry axis, stresses are produced that penetrate into the body of the shell without decaying. These stresses can be balanced by using the solution of the problem of the tension of a shell by concentrated forces applied at the poles $\vartheta = 0$ and $\vartheta = \pi$ [4 and 5].

4. Let us consider the derivation of approximate theories intended for taking care of stresses on the spherical parts of the boundary. The characteristic feature of this approach consists in the fact that the corresponding equations of the approximate theory are constructed by an individual method according to the quantity to be considered. We will illustrate the construction of such a theory for the determination of the displacements u_r and u_α of the points of the middle surface.

We will consider the case when the spherical part of the boundary is subjected to tractions of the form

$$\begin{aligned} \sigma_r &= K_1 P_n(\alpha), \quad \tau_{r\alpha} = L_1 \cosh\alpha \frac{dP_n}{d\alpha} \quad \text{for } r = a \\ \sigma_r &= K_2 P_n(\alpha), \quad \tau_{r\alpha} = L_2 \cosh\alpha \frac{dP_n}{d\alpha} \quad \text{for } r = b \end{aligned} \tag{4.1}$$

where P_n satisfies the Legendre equation

$$\cosh^2\alpha \frac{d^2P_n}{d\alpha^2} + n(n+1)P_n = 0 \tag{4.2}$$

In the case of a closed spherical shell the displacements are

$$u_r = \frac{2(1+\nu)}{E}\Big\{-\frac{(n^2-n-2)+4\nu(n+1)}{4(1-\nu)(2n+3)} r^{n+1}C_1 + \frac{(n^2+3n)-4\nu n}{4(1-\nu)(2n-1)} r^{-n}C_2 + nr^{n-1}C_3 - (n+1)r^{-(n+2)}C_4\Big\}P_n \tag{4.3}$$

$$u_\alpha = \frac{2(1+\nu)}{E}\Big\{-\frac{(n+5)-4\nu}{4(1-\nu)(2n+3)} r^{n+1}C_1 + \frac{(4-n)-4\nu}{4(1-\nu)(2n-1)} r^{-n}C_2 + r^{n-1}C_3 + r^{-(n+2)}C_4\Big\}\cosh\alpha \frac{dP_n}{d\alpha} \tag{4.4}$$

The quantities C_i are found from the boundary conditions (4.1)

$$C_i = \Delta_i / \Delta \tag{4.5}$$

Let us expand all the determinants Δ and Δ_i in series of powers of $\epsilon = h/r_0$, where r_0 is the radius of the middle surface of the shell and $2h$ is its thickness. One finds

$$\Delta = (1 - \nu^2)\,\varepsilon^2 + {}^1/_3\,\{[(n^2 + n) + 1]^2 - \nu^2\,[4\,(n^2 + n) + 1]\}\,\varepsilon^4 + \ldots \tag{4.6}$$

$$\Delta_1 = \frac{1}{2\,(2n+1)}\,r_0^{-n}\,\langle\{(K_2 - K_1)\,[(n + 1) - \nu] + (L_2 - L_1)\,n\,(1 + \nu)\}\,\varepsilon +$$
$$+ \{(K_2 + K_1)\,[(2n + 3) - \nu\,(n + 3)] - (L_2 + L_1)\,n\,[(n^2 + 2n - 1) - \nu\,(n + 3)]\}\,\varepsilon^2 +$$
$$+ {}^1/_6\,\{(K_2 - K_1)\,[(5n^3 + 8n^2 + 11n + 18) - \nu\,(7n^2 + 17n + 18)] - (L_2 - L_1)\,n\,[(2n^3 +$$
$$+ 11n^2 + 23n + 6) - \nu\,(7n^2 + 17n + 18)]\}\,\varepsilon^3 + {}^1/_6\,\{(K_2 + K_1)\,[(2n^4 + 12n^3 +$$
$$+ 17n^2 + 11n + 6) - \nu\,(5n^3 + 20n^2 + 17n + 6)] - (L_2 + L_1)\,n\,[(3n^4 + 11n^3 + 13n^2 +$$
$$+ 15n + 6) - \nu\,(5n^3 + 20n^2 + 17n + 6)]\}\,\varepsilon^4 + \ldots\rangle \tag{4.7}$$

$$\Delta_2 = \frac{1}{2\,(2n+1)}\,r_0^{n+1}\,\langle\{(K_2 - K_1)\,(n + \nu) + (L_2 - L_1)\,(n + 1)\,(1 + \nu)\}\,\varepsilon +$$
$$+ \{(K_2 + K_1)\,[(2n - 1) - \nu\,(n - 2)] - (L_2 + L_1)\,(n + 1)\,[(n^2 - 2) + \nu\,(n - 2)]\}\,\varepsilon^2 +$$
$$+ {}^1/_6\,\{(K_2 - K_1)\,[(5n^3 + 7n^2 + 10n - 10) + \nu\,(7n^2 - 3n + 8)] + (L_2 - L_1)\,(n + 1)\,[(2n^3 -$$
$$- 5n^2 + 7n + 8) + \nu\,(7n^2 - 3n + 8)]\}\,\varepsilon^3 + {}^1/_6\,\{(K_2 + K_1)\,[(-2n^4 + 4n^3 + 7n^2 +$$
$$+ 5n - 2) + \nu\,(-5n^3 + 5n^2 + 8n + 4)] - (L_2 + L_1)\,(n + 1)\,[(3n^4 + n^3 - 2n^2 -$$
$$- 10n - 4) + \nu\,(5n^3 - 5n^2 - 8n - 4)]\}\,\varepsilon^4 + \ldots\rangle \tag{4.8}$$

$$\Delta_3 = \frac{1}{8\,(n-1)\,(2n-1)\,(2n+1)}\,r_0^{-n+4}\,\langle\{(K_2 - K_1)\,[(n^3 + n^2 - n + 1) - \nu(n^2 - 1) - 2\nu^2] +$$
$$+ (L_2 - L_1)\,(n + 1)\,[(n^2 - n + 2) + \nu\,(n^2 - n) - 2\nu^2]\}\,\varepsilon + \{(K_2 + K_1)\,[(2n^3 + 3n^2 - 1) -$$
$$- \nu\,(n^3 + 3n^2 - n - 3) - 2\nu^2\,(n + 1)] - (L_2 + L_1)\,(n + 1)\,[(n^4 + n^3 - 3n^2 - n - 2) -$$
$$- \nu\,(n^3 + 2n^2 - 3n) + 2\nu^2\,(n + 1)]\}\,\varepsilon^2 + {}^1/_6\,\{(K_2 - K_1)\,[(5n^5 + 20n^4 + 16n^3 +$$
$$+ 6n^2 + 21n - 20) - \nu\,(7n^4 + 25n^3 + 5n^2 - 25n - 12) - 2\nu^2\,(7n^2 + 21n - 4)] -$$
$$- (L_2 - L_1)\,(n + 1)\,[(2n^5 + 5n^4 - 4n^3 - 13n^2 - 46n + 8) - \nu\,(7n^4 + 18n^3 - 13n^2 -$$
$$- 12n) + 2\nu^2(7n^2 + 21n - 4)]\}\,\varepsilon^3 + {}^1/_6\,\{(K_2 + K_1)\,[(2n^6 + 16n^5 + 43n^4 +$$
$$+ 25n^3 - 13n^2 - 5n + 4) - \nu\,(5n^5 + 28n^4 + 30n^3 - 40n^2 - 35n + 12) -$$
$$- 2\nu^2\,(5n^3 + 22n^2 + 13n - 4)] - (L_2 + L_1)\,(n + 1)\,[(3n^6 + 12n^5 - 8n^4 -$$
$$- 42n^3 - 15n^2 - 30n + 8) - \nu\,(5n^5 + 23n^4 + 7n^3 - 47n^2 + 12n) +$$
$$+ 2\nu^2\,(5n^3 + 22n^2 + 13n - 4)]\}\,\varepsilon^4 + \ldots\rangle \tag{4.9}$$

$$\Delta_4 = -\frac{1}{8\,(n+2)\,(2n+1)\,(2n+3)}\,r_0^{n+3}\,\langle\{(K_2 - K_1)\,[(-n^3 - 2n^2 + 2) - \nu\,(n^2 + 2n) -$$
$$- 2\nu^2] - (L_2 - L_1)\,n\,[(n^2 + 3n + 4) + \nu\,(n^2 + 3n + 2) - 2\nu^2]\}\,\varepsilon - \{(K_2 + K_1)\,[(2n^3 +$$
$$+ 3n^2) - \nu\,(n^3 - 4n) - 2\nu^2 n] - (L_2 + L_1)\,n\,[(n^4 + 3n^3 - 4n - 4) + \nu\,(n^3 + n^2 -$$
$$- 4n - 4) - 2\nu^2 n]\}\,\varepsilon^2 - {}^1/_6\,\{(K_2 - K_1)\,[(5n^5 + 5n^4 - 14n^3 - 28n^2 + 2n +$$
$$+ 36) + \nu\,(7n^4 + 3n^3 - 28n^2 - 12n) + 2\nu^2\,(7n^2 - 7n - 18)] - (L_2 - L_1)\,n\,[(-2n^5 - 5n^4 +$$
$$+ 4n^3 + 9n^2 + 42n + 48) - \nu\,(7n^4 + 10n^3 - 25n^2 - 40n - 12) + 2\nu^2\,(7n^2 - 7n -$$
$$- 18)]\}\,\varepsilon^3 + {}^1/_6\,\{(K_2 + K_1)\,[(2n^6 - 4n^5 - 7n^4 + 27n^3 + 40n^2 + 8n) +$$
$$+ \nu\,(5n^5 - 3n^4 - 32n^3 + 12n^2 + 48n) + 2\nu^2\,(5n^3 - 7n^2 - 16n)] +$$
$$+ (L_2 + L_1)\,n\,[(3n^6 + 6n^5 - 23n^4 - 50n^3 - 12n^2 + 52n + 48) +$$
$$+ \nu\,(5n^5 + 2n^4 - 35n^3 - 20n^2 + 60n + 48) - 2\nu^2\,(5n^3 - 7n^2 - 16n)]\}\,\varepsilon^4 + \ldots\rangle \tag{4.10}$$

Substituting (4.5) into (4.3) and (4.4), we find

$$\Delta u_r = \frac{2\,(1+\nu)}{E}\left\{-\frac{(n^2 - n - 2) + 4\nu\,(n+1)}{4\,(1-\nu)\,(2n+3)}\,r^{n+1}\Delta_1 + \frac{(n^2 + 3n) - 4\nu n}{4\,(1-\nu)\,(2n-1)}\,r^{-n}\Delta_2 +\right.$$
$$\left. + n r^{n-1}\Delta_3 - (n+1)\,r^{-(n+2)}\Delta_4\right\} P_n \tag{4.11}$$

$$\Delta u_\alpha = \frac{2(1+\nu)}{E}\Big\{-\frac{(n+5)-4\nu}{4(1-\nu)(2n+3)} r^{n+1}\Delta_1 + \frac{(4-n)-4\nu}{4(1-\nu)(2n-1)} r^{-n}\Delta_2 + $$
$$+ r^{n-1}\Delta_3 + r^{-(n+2)}\Delta_4\Big\}\cosh\alpha\,\frac{dP_n}{d\alpha} \quad (4.12)$$

Expressions (4.11) and (4.12) can be used for constructing approximate theories intended for equilibrating the stresses on Γ_2. These expressions show that u_r and u_α could be obtained by satisfying the following system of ordinary differential equations

$$-4\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}+2\right)\Big\langle(1-\nu^2)\varepsilon^2 + {}^1/_3\Big\{\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}\right) -$$
$$-2\cosh^2\alpha\frac{d^2}{d\alpha^2}+1\Big]+\nu^2\Big[4\cosh^2\alpha\frac{d^2}{d\alpha^2}-1\Big]\Big\}\varepsilon^4+\ldots\Big\rangle\frac{E}{1+\nu}\frac{1}{r_0}u_r =$$
$$=\Big\langle\Big\langle 2\Big\langle-\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}+1-\nu\Big](K_2-K_1)-\cosh^2\alpha\frac{d^2}{d\alpha^2}(1-\nu^2)(L_2-L_1)\Big\rangle\varepsilon +$$
$$+2\Big\langle\Big[-2\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}+1\right)+3\nu\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}+2\right)-\nu^2\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}+4\right)\Big]\times$$
$$\times(K_2+K_1)-2\Big\{\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}\right)+4\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big]-\nu\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}\times$$
$$\times\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}\right)+2\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big]-2\nu^2\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big\}(L_2+L_1)\Big\rangle\varepsilon^2+$$
$$+{}^1/_6\Big\langle\Big\{2\Big[8\cosh^2\alpha\frac{d^2}{d\alpha^2}\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}\right)+\cosh^2\alpha\frac{d^2}{d\alpha^2}-6\Big]-2\nu\Big[5\cosh^2\alpha\frac{d^2}{d\alpha^2}\times$$
$$\times\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}\right)-8\cosh^2\alpha\frac{d^2}{d\alpha^2}-36\Big]-6\nu^2\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}+10\Big]\Big\}(K_2-K_1)-$$
$$-\Big\{4\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}\right)+14\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big]-30\nu\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}\right)+$$
$$+2\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big]+2\nu^2\Big[7\cosh^2\alpha\frac{d^2}{d\alpha^2}\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}\right)-10\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big]\Big\}(L_2-L_1)\Big\rangle\varepsilon^3+$$
$$+{}^1/_6\Big\langle\Big\{4\Big[7\cosh^2\alpha\frac{d^2}{d\alpha^2}\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}\right)+4\cosh^2\alpha\frac{d^2}{d\alpha^2}-2\Big]-$$
$$-4\nu\Big[7\cosh^2\alpha\frac{d^2}{d\alpha^2}\left(\cosh\alpha\frac{d^2}{d\alpha^2}\right)+11\cosh^2\alpha\frac{d^2}{d\alpha^2}-6\Big]+2\nu^2\Big[5\cosh^2\alpha\frac{d^2}{d\alpha^2}\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}\right)+$$
$$+24\cosh^2\alpha\frac{d^2}{d\alpha^2}-8\Big]\Big\}(K_2+K_1)-\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big\{4\Big[-2\cosh^2\alpha\frac{d^2}{d\alpha^2}\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}\right)-$$
$$-9\cosh^2\alpha\frac{d^2}{d\alpha^2}+8\Big]+6\nu\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}\left(\cosh^2\alpha\frac{d^2}{d\alpha^2}\right)-2\cosh^2\alpha\frac{d^2}{d\alpha^2}-8\Big]+$$
$$+4\nu^2\Big[5\cosh^2\alpha\frac{d^2}{d\alpha^2}-8\Big]\Big\}(L_2+L_1)\Big\rangle\varepsilon^4+\ldots\Big\rangle\Big\rangle P_n \quad (4.13)$$

$$-4\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}+2\Big]\Big\langle(1-\nu^2)\,\varepsilon^2+{}^1/_3\Big\{\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big(\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big)-$$

$$-2\cosh^2\alpha\frac{d^2}{d\alpha^2}+1\Big]+\nu^2\Big[4\cosh^2\alpha\frac{d^2}{d\alpha^2}-1\Big]\Big\}\,\varepsilon^4+\ldots\Big\rangle\frac{E}{1+\nu}\frac{1}{r_0}u_\alpha=$$

$$=\Big\langle\Big\langle\{2(1-\nu^2)(K_2-K_1)+4(1-\nu^2)(L_2-L_1)\}\,\varepsilon+\Big\{4(1-\nu^2)(K_2+K_1)-$$

$$-2\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}+6\Big](1-\nu^2)(L_2+L_1)\Big\}\,\varepsilon^2+{}^1/_3\Big\langle\Big\{-2\Big(5\cosh^2\alpha\frac{d^2}{d\alpha^2}-2\Big)+$$

$$+3\nu\Big(\cosh^2\alpha\frac{d^2}{d\alpha^2}+2\Big)+\nu^2\Big(7\cosh^2\alpha\frac{d^2}{d\alpha^2}-10\Big)\Big\}(K_2-K_1)+$$

$$+2\Big\{\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big(\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big)+20\Big]-\nu\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big(\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big)+2\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big]+$$

$$+2\nu^2\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}-10\Big]\Big\}(L_2-L_1)\Big\rangle\,\varepsilon^3+{}^1/_6\Big\langle\Big\{-2\Big[19\cosh^2\alpha\frac{d^2}{d\alpha^2}+2\Big]+$$

$$+2\nu\Big[2\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big(\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big)+13\cosh^2\alpha\frac{d^2}{d\alpha^2}+18\Big]+4\nu^2\Big[5\cosh^2\alpha\frac{d^2}{d\alpha^2}-8\Big]\Big\}\times$$

$$\times(K_2+K_1)+\Big\{-48\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}-1\Big]-2\nu\Big[\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big(\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big)+$$

$$+2\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big]-2\nu^2\Big[5\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big(\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big)+34\cosh^2\alpha\frac{d^2}{d\alpha^2}-24\Big\}\times$$

$$\times(L_2+L_1)\Big\rangle\,\varepsilon^4+\ldots\Big\rangle\Big\rangle\cosh\alpha\frac{dP_n}{d\alpha} \qquad (4.14)$$

In this way an approximate theory can be obtained that has an error of arbitrary degree of smallness with respect to ε. If u_r and u_α are to be considered at points not on the middle surface, but in an arbitrary layer, the corresponding value of r must be substituted into Formulas (4.11) and (4.12) to yield new equations for the approximate determination of u_r and u_α. It is not difficult to obtain the equations for arbitrary characteristics of the state of stress in the shell.

5. Here we will consider some of the existing approximate theories of designing spherical shells corresponding to the results obtained above. In the first place we will examine the boundary effects. As an example we will analyze the Vlasov theory [6], which in the case of flexure gives Equation

$$\Big\{(\nabla^2+1)^2+\frac{12(1-\nu^2)R^2}{h^2}\Big\}w=0 \qquad \Big(\nabla^2=\cosh^2\alpha\frac{d^2}{d\alpha^2}\Big) \qquad (5.1)$$

where h is the thickness of the shell. Equation (5.1) can be put in the form

$$(\nabla^2-\mu_1^2)(\nabla^2-\mu_2^2)\,w=0 \qquad (5.2)$$

where

$$\mu_1=-1+\Big(-\frac{12(1-\nu^2)R^2}{h^2}\Big)^{1/2}=\frac{1}{\gamma}\sqrt{-12(1-\nu^2)}-$$

$$-[1+{}^1/_2\sqrt{-12(1-\nu^2)}]+{}^1/_{12}\sqrt{-12(1-\nu^2)}\,\gamma+\ldots$$

$$\mu_2=-1-\Big(-\frac{12(1-\nu^2)R^2}{h^2}\Big)^{1/2}=-\frac{1}{\gamma}\sqrt{-12(1-\nu^2)}- \qquad (5.3)$$

$$-[1-{}^1/_2\sqrt{-12(1-\nu^2)}]-{}^1/_{12}\sqrt{-12(1-\nu^2)}\,\gamma\ldots$$

Asymptotic integration of Equation (5.2) shows us that its solution has the form

$$w = \exp\left\{ \tan^{-1} (\sinh\alpha) \frac{a_{-1}}{\sqrt{\gamma}} i + {}^1/_2 \cosh\alpha + \right.$$

$$\left. + \frac{1}{8a_{-1}} \left[-2 \tan^{-1} (\sinh\alpha)(a_{-1}{}^2 + 2) - \left(\frac{\sinh 2\alpha}{2} + \alpha + \frac{\sinh^3\alpha}{3} \right) \right] \sqrt{\gamma} + \ldots \right\} \tag{5.4}$$

By comparison of the results we see that the Vlasov theory, as is to be expected, also gives an error term of order $\gamma^{\frac{1}{2}}$ in the expansion of the boundary effect. Without going into details, we point out that the same conclusion can be drawn concerning all known technical theories based on the Kirchhoff hypothesis. Now let us turn to the analysis of the accuracy of the technical theories when they are regarded as a means of equilibrating the stresses on the spherical part of the boundary.

In the Vlasov theory, the equations for the determination of u_r and u_α are

$$\left\{ \left[\cosh^2\alpha \frac{d^2}{d\alpha^2} + 2 \right] + \frac{1}{3(1-\nu^2)} \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} + 1 \right)^2 \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} + 2 \right) \varepsilon^2 + \ldots \right\} u_r =$$

$$= \frac{r_0}{2E} \left\langle \left\{ (1+\nu)\cosh^2\alpha \frac{d^2}{d\alpha^2} L + \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} + 1 - \nu \right) K \right\} \frac{1}{\varepsilon} - \right.$$

$$\left. - \left\{ {}^1/_3 \cosh^2\alpha \frac{d^2}{d\alpha^2} \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} \right) L \right\} \varepsilon \right\rangle P_n$$

$$\left\{ -(1+\nu) \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} + 2 \right) + {}^1/_3 \left[\cosh^2\alpha \frac{d^2}{d\alpha^2} \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} + 2 \right) + \right. \right.$$

$$\left. + \frac{1}{1-\nu^2} \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} + 1 \right)^2 \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} + 2 \right) \right] \varepsilon^2 - \frac{1}{3(1-\nu)} \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} + 1 \right)^2 \times$$

$$\left. \times \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} + 2 \right) \varepsilon^4 \right\} u_\alpha = \frac{r_0}{2E} \left\langle \left\{ 2(1+\nu)\cosh^2\alpha \frac{d^2}{d\alpha^2} L + (1+\nu)\cosh^2\alpha \frac{d^2}{d\alpha^2} K \right\} \frac{1}{\varepsilon} - \right.$$

$$- {}^1/_3 \left\{ \left[2\nu\cosh^2\alpha \frac{d^2}{d\alpha^2} \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} \right) + (6-\nu)\,\nu\cosh^2\alpha \frac{d^2}{d\alpha^2} + 2\nu^2 \right] K + \right.$$

$$\left. + \left[\cosh^2\alpha \frac{d^2}{d\alpha^2} \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} \right) - (1-\nu)\cosh^2\alpha \frac{d^2}{d\alpha^2} - 2(1-\nu) \right] L \right\} \varepsilon - {}^1/_3 \left\{ \left[\left(\cosh^2\alpha \frac{d^2}{d\alpha^2} + \right. \right. \right.$$

$$\left. \left. \left. + 1 - \nu \right) \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} + 2 \right) - 2\cosh^2\alpha \frac{d^2}{d\alpha^2} \right] \cosh^2\alpha \frac{d^2}{d\alpha^2} \left(\cosh^2\alpha \frac{d^2}{d\alpha^2} \right) L \right\} \varepsilon^3 \right\rangle \cosh\alpha \frac{dP_n}{d\alpha}$$

Comparison of these relations with the exact expansions (4.13) and (4.14) shows that only the first terms agree.

6. Now let us consider in detail the question of complete equilibrating the stresses on the transverse edges Γ_1 . To begin with, we will assume that the middle surface is a sphere with one circular cutout (Fig.3). Let the traction on the edge $\alpha = \alpha_1$ be

$$\sigma_\alpha = f_1(r), \qquad \tau_{r\alpha} = f_2(r) \tag{6.1}$$

which satisfy the equilibrium condition

$$\int_a^b \left[f_1(r) \frac{1}{\cosh\alpha} + f_2(r) \tanh\alpha \right] r\,dr = 0 \tag{6.2}$$

Making use of Lagrange's principle of virtual displacements, the solution is found in the form

$$
\begin{aligned}
u_r &= \sum_{k=1}^{2} A_k u_{rk}^{(0)} + \sum_{i=1}^{\infty} B_i u_{ri}^{(c)} + \sum_{i=1}^{\infty} D_i u_{ri}^{*(c)} \\
u_\alpha &= \sum_{k=1}^{2} A_k u_{\alpha k}^{(0)} + \sum_{i=1}^{\infty} B_i u_{\alpha i}^{(c)} + \sum_{i=1}^{\infty} D_i u_{\alpha i}^{*(c)} \\
\tau_{r\alpha} &= \sum_{k=1}^{2} A_k \tau_{r\alpha k}^{(0)} + \sum_{i=1}^{\infty} B_i \tau_{r\alpha i}^{(c)} + \sum_{i=1}^{\infty} D_i \tau_{r\alpha i}^{*(c)} \\
\sigma_r &= \sum_{k=1}^{2} A_k \sigma_{rk}^{(0)} + \sum_{i=1}^{\infty} B_i \sigma_{ri}^{(c)} + \sum_{i=1}^{\infty} D_i \sigma_{ri}^{*(c)} \\
\sigma_\alpha &= \sum_{k=1}^{2} A_k \sigma_{\alpha k}^{(0)} + \sum_{i=1}^{\infty} B_i \sigma_{\alpha i}^{(c)} + \sum_{i=1}^{\infty} D_i \sigma_{\alpha i}^{*(c)} \\
\sigma_\varphi &= \sum_{k=1}^{2} A_k \sigma_{\varphi k}^{(0)} + \sum_{i=1}^{\infty} B_i \sigma_{\varphi i}^{(c)} + \sum_{i=1}^{\infty} D_i \sigma_{\varphi i}^{*(c)}
\end{aligned}
\tag{6.3}
$$

where $u_{rk}^{(0)}$, $u_{\alpha k}^{(0)}$, $\tau_{r\alpha k}^{(0)}$, $\sigma_{rk}^{(0)}$, $\sigma_{\alpha k}^{(0)}$, $\sigma_{\varphi k}^{(0)}$ are given by Formulas (3.7) to (3.12), $u_{ri}^{(c)}$, $u_{\alpha i}^{(c)}$, $\sigma_{ri}^{(c)}$, $\tau_{r\alpha i}^{(c)}$, $\sigma_{\alpha i}^{(c)}$, $\sigma_{\varphi i}^{(c)}$ by Formulas (3.16) to (3.21), and $u_{ri}^{*(c)}$, $u_{\alpha i}^{*(c)}$, $\tau_{r\alpha i}^{*(c)}$, $\sigma_{ri}^{*(c)}$, $\sigma_{\alpha i}^{*(c)}$, $\sigma_{\varphi i}^{*(c)}$ by Formulas (3.22) to (3.27).

As generalized parameters we use the coefficients A_k, B_i, D_i Since the homogeneous solutions exactly satisfy the equilibrium conditions and the boundary conditions on Γ_2, the principle of virtual displacements assumes the following form

$$
\int_a^b (\sigma_\alpha \delta u_\alpha + \tau_{r\alpha} \delta u_r)\, r\, dr = \int_a^b [f_1(r)\, \delta u_\alpha + f_2(r)\, \delta u_r]\, r\, dr \tag{6.4}
$$

We will express the variation of the displacements by means of δA_k, δB_i and δD_i. Carrying out the integrations and setting the coefficients of independent variations equal to zero, we obtain the following system:

$$
\sum_{k=1}^{2} m_{jk} A_k + \sum_{i=1}^{\infty} n_{ji} B_i + \sum_{i=1}^{\infty} p_{ji} D_i = c_j \qquad (j = 1, 2) \tag{6.5}
$$

$$
\sum_{k=1}^{2} r_{tk} A_k + \sum_{i=1}^{\infty} g_{ti} B_i + \sum_{i=1}^{\infty} h_{ti} D_i = d_t \qquad (t = 1, 2, 3, \ldots \infty) \tag{6.6}
$$

$$
\sum_{k=1}^{2} a_{lk} A_k + \sum_{i=1}^{\infty} b_{li} B_i + \sum_{i=1}^{\infty} q_{li} D_i = s_l \qquad (l = 1, 2, 3, \ldots \infty) \tag{6.7}
$$

where

$$m_{jk} = \int_a^b [\sigma_{\alpha k}{}^{(0)} u_{\alpha j}{}^{(0)} + \tau_{r\alpha k}{}^{(0)} u_{rj}{}^{(0)}]\, r\, dr, \quad n_{ji} = \int_a^b [\sigma_{\alpha i}{}^{(c)} u_{\alpha j}{}^{(0)} + \tau_{r\alpha i}{}^{(c)} u_{rj}{}^{(0)}]\, r\, dr$$

$$p_{ji} = \int_a^b [\sigma_{\alpha i}{}^{*(c)} u_{\alpha j}{}^{(0)} + \tau_{r\alpha i}{}^{*(c)} u_{rj}{}^{(0)}]\, r\, dr, \quad r_{lk} = \int_a^b [\sigma_{\alpha k}{}^{(0)} u_{\alpha l}{}^{(c)} + \tau_{r\alpha k}{}^{(0)} u_{rl}{}^{(c)}]\, r\, dr$$

$$g_{li} = \int_a^b [\sigma_{\alpha i}{}^{(c)} u_{\alpha l}{}^{(c)} + \tau_{r\alpha i}{}^{(c)} u_{rl}{}^{(c)}]\, r\, dr, \quad h_{li} = \int_a^b [\sigma_{\alpha i}{}^{*(c)} u_{\alpha l}{}^{(c)} + \tau_{r\alpha i}{}^{*(c)} u_{rl}{}^{(c)}]\, r\, dr$$

$$a_{lk} = \int_a^b [\sigma_{\alpha k}{}^{(0)} u_{\alpha l}{}^{*(c)} + \tau_{r\alpha k}{}^{(0)} u_{rl}{}^{*(c)}]\, r\, dr \tag{6.8}$$

$$b_{li} = \int_a^b [\sigma_{\alpha i}{}^{(c)} u_{\alpha l}{}^{*(c)} + \tau_{r\alpha i}{}^{(c)} u_{rl}{}^{*(c)}]\, r\, dr$$

$$q_{li} = \int_a^b [\sigma_{\alpha i}{}^{*(c)} u_{\alpha l}{}^{*(c)} + \tau_{r\alpha i}{}^{*(c)} u_{rl}{}^{*(c)}]\, r\, dr, \quad c_j = \int_a^b [f_1(r)\, u_{\alpha j}{}^{(0)} + f_2(r)\, u_{rj}{}^{(0)}]\, r\, dr$$

$$d_l = \int_a^b [f_1(r)\, u_{\alpha l}{}^{(c)} + f_2(r)\, u_{rl}{}^{(c)}]\, r\, dr, \quad s_l = \int_a^b [f_1(r)\, u_{\alpha l}{}^{*(c)} + f_2(r)\, u_{rl}{}^{*(c)}]\, r\, dr$$

In system (6.5) to (6.7), Equation (6.5) corresponds to δA_k, (6.6) to δB_i and (6.7) to δD_i. After solving this system for all the coefficients A_k, B_i and D_i we have the solution of the problem.

7. It can be proved that system (6.5) to (6.7) is positive definite in the energy space H_e and thus is always solvable for physically reasonable restrictions on the functions f_1 and f_2. We will study the structure of this system when the thickness parameter γ tends to zero.

First of all we will clarify the assumption concerning the external loading. Since the stresses σ_α and $\tau_{r\alpha}$, corresponding to the roots of the second group have different orders with respect to γ, namely $\sigma_\alpha \sim 1/\sqrt{\gamma}$ and $\tau_{r\alpha} \sim 1$, it is essential that the order of $f_2(r)$ exceeds that of $f_1(r)$ by a factor $\sqrt{\gamma}$. This case will be examined in the following.

Here we introduce the notation

$$\int_{-1}^{1} f_1(\zeta)\, u_{\alpha k}{}^{(0)}\, d\zeta = F_k \gamma, \qquad \int_{-1}^{1} f_2(\zeta)\, u_{rk}{}^{(0)}\, d\zeta = G_k \gamma$$

$$\int_{-1}^{1} f_1(\zeta)\, u_{\alpha i}{}^{(c)}\, d\zeta = N_i \gamma, \qquad \int_{-1}^{1} f_2(\zeta)\, u_{ri}{}^{(c)}\, d\zeta = M_i \gamma \tag{7.1}$$

$$\int_{-1}^{1} f_1(\zeta)\, u_{\alpha i}{}^{*(c)}\, d\zeta = P_i \gamma, \qquad \int_{-1}^{1} f_2(\zeta)\, u_{ri}{}^{*(c)}\, d\zeta = Q_i \gamma$$

Let us represent all these characteristics in the form of power series in γ. In view of the assumption made about the external loading, we have

Expansions
$$F_k = F_{k0} + F_{k1}\sqrt{\gamma} + F_{k2}\gamma + \dots \quad \text{to } F_k,\ N_i,\ P_i \tag{7.2}$$
$$G_k = G_{k1}\sqrt{\gamma} + G_{k2}\gamma + \dots \quad \text{to } G_k,\ M_i,\ Q_i \tag{7.3}$$

When $\alpha = \alpha_1$, the quantities $m(\alpha)$ and $dm(\alpha)/d\alpha$ can be decomposed in the following way: $m(\alpha_1) = 1$ for the second and third group,

for the roots of the second group
$$\left.\frac{dm}{d\alpha}\right|_{\alpha=\alpha_1} = \frac{1}{\cosh\alpha_1}\frac{a_{-1}}{\sqrt{\gamma}}i + \frac{1}{2}\sinh\alpha_1 -$$
$$-\frac{1}{8a_{-1}}i\,[(1-8a_{-1}a_1) - (\cosh 2\alpha_1 + 1 + \sinh^2\alpha_1\cosh\alpha_1)]\sqrt{\gamma} + \dots \tag{7.4}$$

for the roots of the third group
$$\left.\frac{dm}{d\alpha}\right|_{\alpha=\alpha_1} = \frac{1}{\cosh\alpha_1}\frac{m_{-1}}{\gamma}i + \frac{1}{2}\sinh\alpha_1 -$$
$$-\frac{1}{8m_{-1}}i\,[(1-8m_{-1}m_1) - (\cosh 2\alpha_1 + 1 + \sinh^2\alpha_1\cosh\alpha_1)]\gamma + \dots \tag{7.5}$$

The coefficients of the system then have the form
$$\begin{aligned} A_k &= A_{k0} + A_{k1}\sqrt{\gamma} + A_{k2}\gamma + \dots \\ B_i &= B_{i0} + B_{i1}\sqrt{\gamma} + B_{i2}\gamma + \dots \\ D_i &= D_{i0} + D_{i1}\sqrt{\gamma} + D_{i2}\gamma + \dots \end{aligned} \tag{7.6}$$

Substituting (7.4), (7.5) and (7.6) into (6.5) to (6.7) and taking account of (6.8), we obtain
$$\sum_{k=1}^{2}\left\{\frac{16G}{(1-2\nu)^2}a\sqrt{-1}(1-\nu^2)(1-\nu)(a_{-1}^{(j)})^2 \times\right.$$
$$\left.\times\,[a_{-1}^{(j)} - (a_{-1}^{(k)})]\sqrt{\gamma}(A_{k0} + A_{k1}\sqrt{\gamma} + \dots)\right\} +$$
$$+\sum_{i=1}^{\infty}\left\{-\frac{48G}{(1-2\nu)^2}a\sqrt{-1}(1-\nu^2)\nu\frac{\sinh^2\omega_i}{\omega_i}\sqrt{\gamma}(B_{i0} + B_{i1}\sqrt{\gamma} + B_{i2}\gamma + \dots)\right\} +$$
$$+\sum_{i=1}^{\infty}\left\{-\frac{8G}{(1-2\nu)^2}a\sqrt{-1}(1-\nu^2)\nu\sinh\alpha_1 \times\right.$$
$$\left.\times\frac{\sinh^2\delta_i}{\delta_i}a_{-1}^{(j)}(D_{i0} + D_{i1}\sqrt{\gamma} + D_{i2}\gamma + \dots)\gamma\right\} + \dots = (F_j + G_j)\gamma \quad (j = 1, 2) \tag{7.7}$$

$$\sum_{k=1}^{2}\left\{-\frac{48G}{(1-2\nu)^2}a\sqrt{-1}(1-\nu^2)\nu\frac{\sinh^2\omega_t}{\omega_t}\sqrt{\gamma}(A_{k0} + A_{k1}\sqrt{\gamma} + A_{k2}\gamma + \dots)\right\} +$$
$$+\left\{-\frac{4G}{(1-2\nu)}a\sqrt{-1}\sum_{i=1,\ i\neq t}^{\infty}\frac{\omega_i^2\omega_t^2(\cosh^2\omega_i - \cosh^2\omega_t)}{(\omega_i^2-\omega_t^2)^2(\omega_i-\omega_t)}\times\right.$$

$$\times \left[\frac{2\nu}{1-2\nu}(\omega_i^2+\omega_t^2)+\frac{2(1-\nu)}{1-2\nu}\omega_i\omega_t\right](B_{i0}+B_{i1}\sqrt{\gamma}+B_{i2}\gamma+\dots)+$$

$$+\frac{2G}{(1-2\nu)^2}a\sqrt{-1}\,\omega_t^3\,({}^2/_3\cosh^2\omega_t-1)\,(B_{t0}+B_{t1}\sqrt{\gamma}+B_{t2}\gamma+\dots)\Big\}+\dots$$

$$\dots=N_t\gamma+M_t\gamma^{3/2}+\dots \quad (t=1,2,3\dots\infty) \qquad (7.8)$$

$$\sum_{k=1}^{2}\left\{-\frac{8G}{(1-2\nu)^2}a\sqrt{-1}\,(1-\nu^2)\,\nu\sinh\alpha_1\frac{\sinh^2\delta_l}{\delta_l}a_{-1}^{(k)}(A_{k0}+A_{k1}\sqrt{\gamma}+A_{k2}\gamma+\dots)\right\}+$$

$$+\left\{-\frac{4G}{(1-2\nu)}a\sqrt{-1}\sum_{i=1,\ i\neq l}^{\infty}\frac{\delta_i^2\delta_l^2(\sinh^2\delta_i-\sinh^2\delta_l)}{(\delta_i^2-\delta_l^2)^2(\delta_i-\delta_l)}\right.\times$$

$$\times\left[\frac{2\nu}{1-2\nu}(\delta_i^2+\delta_l^2)+\frac{2(1-\nu)}{1-2\nu}\delta_i\delta_l\right](D_{i0}+D_{i1}\sqrt{\gamma}+D_{i2}\gamma+\dots)+$$

$$+\frac{2G}{(1-2\nu)^2}a\sqrt{-1}\,\delta_l^3\left(\frac{2}{3}\sinh^2\delta_l+1\right)(D_{l0}+D_{l1}\sqrt{\gamma}+D_{l2}\gamma+\dots)\Big\}=$$

$$=P_l\gamma+Q_l\gamma^{3/2}+\dots \quad (l=1,2,3,\dots\infty) \qquad (7.9)$$

Equating to zero the coefficients of the same powers of γ, we obtain

$$B_{t0}=0,\ D_{l0}=0,\ A_{k0}=0,\ B_{t1}=0,\ D_{l1}=0,\ A_{k1}\neq 0,\ B_{t2}\neq 0,\ D_{l2}\neq 0$$

It is clear that the coefficients A_{k1}, B_{t2} and D_{l2} can be determined independent of each other, i.e. A_{k1} $(k=1,2)$ can be found from two equations, and B_{t2} D_{l_2} from an infinite system

$$(j=1,2)$$

$$\sum_{k=1}^{2}\left\{\frac{16G}{(1-2\nu)^3}\sqrt{-1}\,a\,(1-\nu^2)(1-\nu)(a_{-1}^{(j)})^2[a_{-1}^{(j)}-a_{-1}^{(k)}]\right\}A_{k1}=F_j+G_j$$

$$-\frac{4Ga\sqrt{-1}}{(1-2\nu)}a\sqrt{-1}\sum_{i=1,\ i\neq t}^{\infty}\frac{\omega_i^2\omega_t^2(\cosh^2\omega_i-\cosh^2\omega_t)}{(\omega_i^2-\omega_t^2)^2(\omega_i-\omega_t)}\left[\frac{2\nu}{1-2\nu}(\omega_i^2+\omega_t^2)+\right.$$

$$\left.+\frac{2(1-\nu)}{1-2\nu}\omega_i\omega_t\right]B_{i2}+\frac{2G}{(1-2\nu)^2}a\sqrt{-1}\,\omega_t^3\left(\frac{2}{3}\cosh^2\omega_t-1\right)B_{t2}=$$

$$=N_t+\frac{48G}{(1-2\nu)^2}a(1-\nu^2)\,\nu\frac{\sinh^2\omega_t}{\omega_t}\sqrt{-1}\sum_{k=1}^{2}A_{k1} \qquad (t=1,2,\dots)$$

$$-\frac{4G}{(1-2\nu)}a\sqrt{-1}\sum_{i=1,\ i\neq l}^{\infty}\frac{\delta_i^2\delta_l^2(\sinh^2\delta_i-\sinh^2\delta_l)}{(\delta_i^2-\delta_l^2)^2(\delta_i-\delta_l)}\left[\frac{2\nu}{1-2\nu}(\delta_i^2+\delta_l^2)+\frac{2(1-\nu)}{1-2\nu}\delta_i\delta_l\right]\times$$

$$\times D_{i2}+\frac{2G}{(1-2\nu)^2}a\sqrt{-1}\,\delta_l^3\left(\frac{2}{3}\sinh^2\delta_l+1\right)D_{l2}=P_l \qquad (l=1,2,\dots)$$

By continuing the process of asymptotic decomposition of the system, we can calculate A_{k2}, B_{l3}, D_{l3} etc.

It is important to note that the matrices of the infinite system for B_{l2}, and D_{l2} are identical with the matrices obtained in the problems of flexure and tension of thick plates. The inversion of these flexure matrices has been carried out by the method of reduction, and the coefficients B_{l2}, D_{l2} have been found to the necessary degree of accuracy.

8. Under the above assumptions concerning the external loading, the coefficients B_i and D_i have order of smallness $\sqrt{\gamma}$ times greater than A_k. The actual expansions of the coefficients have the forms

$$A_k = A_{k1}\sqrt{\gamma} + A_{k2}\gamma + \dots, \qquad B_i = B_{i2}\gamma + B_{i3}\gamma^{3/2} + \dots,$$
$$D_i = D_{i2}\gamma + D_{i3}\gamma^{3/2} + \dots$$

The general solution of the problem of determining the states of stress and strain in a shell can be found by means of superposition of the solutions corresponding to the different groups of roots

$$u_r = \frac{1}{1-2\nu}\Big\{\sum_{k=1}^{2} 2\,(a_{-1}^{(k)})^2 (1-\nu)\exp\Big[\frac{a_{-1}^{(k)}}{\sqrt{\gamma}} + \dots\Big] A_{k1} + \sum_{i=1}^{\infty}\langle[2(1-\nu)\cosh\omega_i + \omega_i\sinh\omega_i]\cosh\omega_i\zeta - \zeta\omega_i\cosh\omega_i\sinh\omega_i\zeta\rangle \times B_{i2}\exp\Big[\frac{2\omega_i}{\gamma} + \dots\Big]\gamma + \sum_{i=1}^{\infty}\langle[2(1-\nu)\sinh\delta_i + \delta_i\cosh\delta_i]\sinh\delta_i\zeta - \zeta\delta_i\sinh\delta_i\cosh\delta_i\zeta\rangle D_{i2}\exp\Big[\frac{2\delta_i}{\gamma} + \dots\Big]\gamma + \dots\Big\} \tag{8.1}$$

$$u_\alpha = -\frac{1}{1-2\nu}\Big\{\sum_{k=1}^{2}[(a_{-1}^{(k)})^2(1-\nu)\zeta - 2(1-\nu^2)]\exp\Big[\frac{a_{-1}^{(k)}}{\sqrt{\gamma}} + \dots\Big]A_{k1}\sqrt{\gamma} + \sum_{i=1}^{\infty}\Big\langle\Big[(1-2\nu)\frac{\cosh\omega_i}{\omega_i} - \sinh\omega_i\Big]\sinh\omega_i\zeta + \zeta\cosh\omega_i\cosh\omega_i\zeta\Big\rangle \times B_{i2}\exp\Big[\frac{2\omega_i}{\gamma} + \dots\Big]\gamma + \sum_{i=1}^{\infty}\Big\langle\Big[(1-2\nu)\frac{\sinh\delta_i}{\delta_i} - \cosh\delta_i\Big]\cosh\delta_i\zeta + \zeta\sinh\delta_i\sinh\delta_i\zeta\Big\rangle D_{i2}\exp\Big[\frac{2\delta_i}{\gamma} + \dots\Big]\gamma + \dots\Big\} \tag{8.2}$$

$$\sigma_\alpha = \frac{2G}{1-2\nu}a^{-1}\Big\{\sum_{k=1}^{2} 12(1-\nu^2)\zeta A_{k1}\exp\Big[\frac{a_{-1}^{(k)}}{\sqrt{\gamma}} + \dots\Big] + 2\sum_{i=1}^{\infty}\Big\langle\Big(\frac{\cosh\omega_i}{\omega_i} - \sinh\omega_i\Big)\sinh\omega_i\zeta + \zeta\cosh\omega_i\cosh\omega_i\zeta\Big\rangle B_{i2}\exp\Big[\frac{2\omega_i}{\gamma} + \dots\Big] + 2\sum_{i=1}^{\infty}\Big\langle\Big(\frac{\sinh\delta_i}{\delta_i} - \cosh\delta_i\Big)\cosh\delta_i\zeta + \zeta\sinh\delta_i\sinh\delta_i\zeta\Big\rangle D_{i2}\exp\Big[\frac{2\delta_i}{\gamma} + \dots\Big]\Big\} \tag{8.3}$$

$$\tau_{r\alpha} = \frac{2G}{1-2\nu} a^{-1} \left\{ \sum_{k=1}^{2} 3(1-\nu^2)(\zeta^2-1) \exp\left[\frac{a_{-1}^{(k)}}{\sqrt{\gamma}} + \ldots\right] A_{k1} \sqrt{\gamma} + \right.$$
$$+ \sum_{i=1}^{\infty} [\sinh\omega_i \cosh\omega_i\zeta - \zeta \cosh'\omega_i \sinh\omega_i\zeta]\, \omega_i B_{i2} \exp\left[\frac{2\omega_i}{\gamma} + \ldots\right] +$$
$$\left. + \sum_{i=1}^{\infty} [\cosh\delta_i \sinh\delta_i\zeta - \zeta \sinh\delta_i \cosh\delta_i\zeta]\, \delta_i D_{i2} \exp\left[\frac{2\delta_i}{\gamma} + \ldots\right] + \ldots \right\} \qquad (8.4)$$

In Formulas (8.1) to (8.4), the first terms of the right-hand sides correspond to the technical theory of shells. The subsequent terms are supplementary to the solution of the Kirchhoff theory. On the part of the boundary where $\alpha = \alpha_1$, the supplementary terms in σ_α have exactly the smae order as those in the technical theory. Moreover, the supplementary terms in the stress $\tau_{r\alpha}$ become of basic inportance as $\gamma \to 0$.

9. All the preceding results refer to shells with a spherical middle surface and one cutout. When the middle surface is still spherical but there are two cutouts at the poles, the general solution (6.3) must be supplemented with the solution for the tension of a shell by two concentrated forces, which was mentioned at the end of Section 3. However, the state of stress in the vicinity of each cutout can be found as described above.

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